

HOMOGENIZATION OF RANDOM FRACTIONAL OBSTACLE PROBLEMS VIA Γ -CONVERGENCE

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ABSTRACT. Γ -convergence methods are used to prove homogenization results for fractional obstacle problems in periodically perforated domains. The obstacles have random sizes and shapes and their capacity scales according to a stationary ergodic process. We use a trace-like representation of fractional Sobolev norms in terms of weighted Sobolev energies established in [8], a weighted ergodic theorem and a joining lemma in varying domains following the approach by [1].

Our proof is alternative to those contained in [6], [7].

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1. INTRODUCTION

The homogenization of (non-)linear elliptic obstacle problems in periodically perforated domains has received much attention after the seminal papers of Marchenko and Khruslov [27], Rauch and Taylor [29], [30] and Cioranescu and Murat [13] (see [10], [3], [20], [18], [14], [15], [16], [26], [1] and [2], [5], [12], [17] for a more exhaustive list of references). The problem has been successfully tackled by making use of abstract techniques of Γ -convergence, and fully solved in a series of papers by Dal Maso [14], [15], [16]. A constructive approach in the periodic case for bilateral obstacles has been developed by Ansini and Braides [1]. In general, a relaxation process takes place and the limit problem contains a finite penalization term related to the capacity of the homogenizing obstacles.

All the quoted results deal with Sobolev type energies and deterministic distributions of the set of obstacles with deterministic sizes and shapes. More recently, two papers [6], [7] have enlarged the stage to fractional Sobolev energies and by considering random sizes and shapes for the obstacles. More precisely, given a probability space $(\Omega, \mathcal{P}, \mathbb{P})$, for all $\omega \in \Omega$ consider a periodic distribution of sets $T_\varepsilon(\omega)$ and let $v_\varepsilon(\cdot, \omega)$ be the solution of the problem

$$\begin{cases} (-\Delta)^s v(y) \geq 0 & y \in \mathbf{R}^{N-1} \\ (-\Delta)^s v(y) = 0 & y \in \mathbf{R}^{N-1} \setminus T_\varepsilon(\omega), \text{ and } y \in T_\varepsilon(\omega) \text{ if } v(y) > \psi(y) \\ v(y) \geq \psi(y) & y \in T_\varepsilon(\omega). \end{cases} \quad (1.1)$$

The operator $(-\Delta)^s$ is the fractional Laplace operator of order $s \in (0, 1)$ defined in terms of the Fourier transform, by $\mathcal{F}((-\Delta)^s v)(\xi) = |\xi|^{2s} \hat{v}(\xi)$; ψ is the *obstacle function* and it is assumed to be in

$C^{1,1}(\mathbf{R}^{N-1})$. In case $s = 1/2$ the minimum problem in (1.1) is known as Signorini's problem and it is related to a semi-permeable membrane model. We refer to the papers [6] and [7] for a more detailed description of the underlying physical model.

Problem (1.1) has a natural variational character. Indeed, it can be interpreted as the Euler-Lagrange equation solved by the minimizer of

$$\inf_{\dot{H}^s(\mathbf{R}^{N-1})} \left\{ \|v\|_{\dot{H}^s(\mathbf{R}^{N-1})} : v \geq \psi \text{ on } T_\varepsilon(\omega) \right\}. \quad (1.2)$$

Here $\|v\|_{\dot{H}^s(\mathbf{R}^{N-1})} = \| |\xi|^{2s} \hat{v}(\xi) \|_{L^2(\mathbf{R}^{N-1})}$ is the usual norm in the homogeneous fractional Sobolev space $\dot{H}^s(\mathbf{R}^{N-1})$.

An additional variational characterization of problem (1.1) can be given by following the work by Caffarelli and Silvestre [8] who have represented fractional Sobolev norms on \mathbf{R}^{N-1} in terms of boundary value problems for degenerate (but local!) elliptic equations in the higher dimensional half-space \mathbf{R}_+^N ; equivalently, in terms of minimal energy extensions of a (suitable) weighted Dirichlet integral as for the harmonic extension of $\dot{H}^{1/2}$ functions. It turns then out that the extension $u_\varepsilon(\cdot, \omega)$ of $v_\varepsilon(\cdot, \omega)$ to \mathbf{R}_+^N solves the problem

$$\inf_{W^{1,2}(\mathbf{R}_+^N, |x_N|^a)} \left\{ \int_{\mathbf{R}_+^N} |x_N|^a |\nabla u(y, x_N)|^2 d\mathcal{L}^N : u(y, 0) \geq \psi(y) \ y \in T_\varepsilon(\omega) \right\}. \quad (1.3)$$

Here, the parameter a ruling the degeneracy of the elliptic equation equals $2s - 1$ (and thus belongs to $(-1, 1)$), and $W^{1,2}(\mathbf{R}_+^N, |x_N|^a)$ is the weighted Sobolev space associated to the measure $|x_N|^a d\mathcal{L}^N(y, x_N)$.

To investigate the asymptotic behaviour of $u_\varepsilon(\cdot, \omega)$ as ε vanishes some assumptions have to be imposed on the obstacles set $T_\varepsilon(\omega)$. Mild hypotheses have been introduced in [6], [7]: the set $T_\varepsilon(\omega)$ is the union of periodically distributed sets (but with random sizes and shapes!) whose capacity scales according to a stationary and ergodic process γ (see (Hp 1) and (Hp 2) in Section 2). Under these assumptions Caffarelli and Mellet [7] have proven that there exists a constant $\alpha_0 \geq 0$ such that the solution $u_\varepsilon(\cdot, \omega)$ of (1.3) converges locally weakly in $W^{1,2}(\mathbf{R}_+^N, |x_N|^a)$ and \mathbb{P} a.s. in Ω to the solution \bar{u} of

$$\inf_{W^{1,2}(\mathbf{R}_+^N, |x_N|^a)} \left\{ \int_{\mathbf{R}_+^N} |x_N|^a |\nabla u(y, x_N)|^2 d\mathcal{L}^N + \frac{\alpha_0}{2} \int_{\mathbf{R}^{N-1}} |(\psi(y) - u(y, 0)) \vee 0|^2 dy \right\}. \quad (1.4)$$

The proof of such a result relies on the regularity of fractional obstacle problems established by Caffarelli et al. [9], and on the PDEs approach to homogenization based on the Tartar's oscillating test function method (see [31], [13] and [12] for further references).

The aim of this paper is to give an alternative elementary proof of the above quoted homogenization results via Γ -convergence techniques. We are able to avoid the use of the regularity theory developed in [9] and thus to relax the smoothness assumption on the obstacle function ψ . In addition, we determine

explicitely the constant α_0 in the capacitary contribution of the limit energy, and show that it equals the expectation of the process $\mathbb{E}[\gamma]$ (see Theorem 2.4).

Despite this, the proof is not self-contained since we still use the trace-like representation for fractional norms established in [8]. A direct approach is still under investigation, and deserves additional efforts since the difficulties introduced in the problem by the non-locality of fractional energies.

The main tools of our analysis are a joining lemma in varying boundary domains for weighted energies and a weighted version of Birkhoff's ergodic theorem. The joining lemma follows the line of the analogous result in perforated open sets for standard Sobolev spaces proved by Ansini and Braides [1]. It is a variant of an idea by De Giorgi [19] in the setting of varying domains, on the way of matching boundary conditions by increasing the energy only up to a small error. This method is elementary and based on a clever slicing and averaging argument, looking for those zones where the energy does not concentrate. The joining lemma allows us to reduce in the Γ -limit process to families of functions which are constants on suitable annuli surrounding the obstacle sets. Thus, to estimate the capacitary contribution close to the obstacle set $T_\varepsilon(\omega)$ we exploit the capacitary scaling assumption on the process γ together with a weighted variant of Birkhoff's ergodic theorem (see Theorem 4.1). This argument allows us to show that α_0 equals $\mathbb{E}[\gamma]$.

In Section 2 we list the assumptions and state the homogenization result. To avoid unnecessary generality we deal with the model case of p -norms, $p \in (1, +\infty)$, since this case contains all the features of the problem. Section 3 collects several results concerning weighted Sobolev spaces in case the weight function is a Muckenhoupt weight of the form $w(y, x_N) = |x_N|^a$. A weighted ergodic theorem relevant in our analysis is proved in Section 4. In Section 5 we prove the Γ -convergence theorem. Finally, in Section 6 we indicate several possible generalizations.

2. STATEMENT OF THE MAIN RESULT

2.1. Basic Notations. The ball in \mathbf{R}^N with centre x and radius $r > 0$ is denoted by $B_r(x)$, and simply by B_r in case $x = \underline{0}$. The interior and the closure of a set $E \subset \mathbf{R}^N$ are denoted by $\text{int}(E)$ and \overline{E} , respectively. Given two sets $E \subset\subset F$ in \mathbf{R}^N , a *cut-off function between E and F* is any $\varphi \in C_0^\infty(F)$ such that $\varphi|_E \equiv 1$.

Not to overburden the notation each set $E \subseteq \mathbf{R}^{N-1}$ and its copy $E \times \{0\} \subseteq \mathbf{R}^N$ will be undistinguished.

In the sequel U denotes any connected open subset of the half-space $\mathbf{R}_+^N := \{x = (y, x_N) : y \in \mathbf{R}^{N-1}, x_N > 0\}$ whose boundary is Lipschitz regular. The part of the boundary of $U \subseteq \mathbf{R}_+^N$ lying on $\{x_N = 0\}$ is denoted by $\partial_N U := \partial U \cap \{x_N = 0\}$.

We use standard notations for Hausdorff and Lebesgue measures, and Lebesgue spaces. The integration with respect to the measure $\mathcal{H}^{N-1} \llcorner \{x_N = 0\}$ is denoted by dy , and for $V \subseteq \{x_N = 0\}$ the spaces $L^p(V, \mathcal{H}^{N-1} \llcorner \{x_N = 0\})$ simply by $L^p(V)$, $p \in [1, +\infty]$.

The lattice in \mathbf{R}^{N-1} underlying the periodic homogenization process is identified via the points $y_j^{\mathbf{i}} := \mathbf{i}\varepsilon_j \in \mathbf{R}^{N-1}$, $x_j^{\mathbf{i}} := (y_j^{\mathbf{i}}, 0) \in \mathbf{R}^N$ and the cubes $Q_j^{\mathbf{i}} := y_j^{\mathbf{i}} + \varepsilon_j[-1/2, 1/2)^{N-1} \subset \mathbf{R}^{N-1}$, $\mathbf{i} \in \mathbf{Z}^{N-1}$. Here, $(\varepsilon_j)_j$ is a positive infinitesimal sequence. Finally, for any set $E \subseteq \mathbf{R}^{N-1}$ define

$$\mathcal{I}_j(E) := \{\mathbf{i} \in \mathbf{Z}^{N-1} : Q_j^{\mathbf{i}} \subseteq E\}.$$

2.2. Γ -convergence. We recall the notion of Γ -convergence introduced by De Giorgi in a generic metric space (X, d) endowed with the topology induced by d (see [17],[4]). A sequence of functionals $F_j : X \rightarrow [0, +\infty]$ Γ -converges to a functional $F : X \rightarrow [0, +\infty]$ in $u \in X$, in short $F(u) = \Gamma\text{-}\lim_j F_j(u)$, if the following two conditions hold:

- (i) (*liminf inequality*) $\forall (u_j)$ converging to u in X , we have $\liminf_j F_j(u_j) \geq F(u)$;
- (ii) (*limsup inequality*) $\exists (u_j)$ converging to u in X such that $\limsup_j F_j(u_j) \leq F(u)$.

We say that F_j Γ -converges to F (or $F = \Gamma\text{-}\lim_j F_j$) if $F(u) = \Gamma\text{-}\lim_j F_j(u) \forall u \in X$. We may also define the *lower* and *upper* Γ -limits as

$$\begin{aligned} \Gamma\text{-}\limsup_j F_j(u) &= \inf\{\limsup_j F_j(u_j) : u_j \rightarrow u\}, \\ \Gamma\text{-}\liminf_j F_j(u) &= \inf\{\liminf_j F_j(u_j) : u_j \rightarrow u\}, \end{aligned}$$

respectively, so that conditions (i) and (ii) are equivalent to $\Gamma\text{-}\limsup_j F_j(u) = \Gamma\text{-}\liminf_j F_j(u) = F(u)$. Moreover, the functions $\Gamma\text{-}\limsup_j F_j$ and $\Gamma\text{-}\liminf_j F_j$ are lower semicontinuous.

One of the main reasons for the introduction of this notion is explained by the following fundamental theorem (see [17, Theorem 7.8]).

Theorem 2.1. *Let $F = \Gamma\text{-}\lim_j F_j$, and assume there exists a compact set $K \subset X$ such that $\inf_X F_j = \inf_K F_j$ for all j . Then there exists $\min_X F = \lim_j \inf_X F_j$. Moreover, if (u_j) is a converging sequence such that $\lim_j F_j(u_j) = \lim_j \inf_X F_j$ then its limit is a minimum point for F .*

2.3. Assumptions and Statement of the Main Result. We consider a probability space $(\Omega, \mathcal{P}, \mathbb{P})$. For all $\omega \in \Omega$ and $j \in \mathbf{N}$ the set $T_j(\omega) \subseteq \mathbf{R}^{N-1}$ is given by

$$T_j(\omega) = \cup_{\mathbf{i} \in \mathbf{Z}^{N-1}} T_j^{\mathbf{i}}(\omega)$$

where the sets $T_j^{\mathbf{i}}(\omega) \subseteq Q_j^{\mathbf{i}}$ satisfy the following conditions:

(Hp 1). *Capacitary Scaling:* There exist a positive infinitesimal sequence $(\delta_j)_j$ and a process $\gamma : \mathbf{Z}^{N-1} \times \Omega \rightarrow [0, +\infty)$ such that for all $\mathbf{i} \in \mathbf{Z}^{N-1}$ and $\omega \in \Omega$

$$\text{cap}_{p,\mu}(T_j^{\mathbf{i}}(\omega)) = \delta_j \gamma(\mathbf{i}, \omega).$$

(Hp 2). *Ergodicity & Stationarity of the Process:* The process $\gamma : \mathbf{Z}^{N-1} \times \Omega \rightarrow [0, +\infty)$ is stationary ergodic: There exists a family of measure-preserving transformations $\tau_{\mathbf{k}} : \Omega \rightarrow \Omega$ satisfying for all $\mathbf{i}, \mathbf{k} \in \mathbf{Z}^{N-1}$ and $\omega \in \Omega$

$$\gamma(\mathbf{i} + \mathbf{k}, \omega) = \gamma(\mathbf{i}, \tau_{\mathbf{k}}\omega), \tag{2.1}$$

and such that if $A \subseteq \Omega$ is an invariant set, i.e. $\tau_{\mathbf{k}}A = A$ for all $\mathbf{k} \in \mathbf{Z}^{N-1}$, then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

Moreover, for some $\gamma_0 > 0$ we have for all $\mathbf{i} \in \mathbf{Z}^{N-1}$ and \mathbb{P} a.s. $\omega \in \Omega$

$$\gamma(\mathbf{i}, \omega) \leq \gamma_0.$$

(Hp 3). *Strong Separation:* There exist $\varepsilon, M > 0$ such that for all $\mathbf{i} \in \mathbf{Z}^{N-1}$, $\omega \in \Omega$, and for every $\varepsilon_j \in (0, \varepsilon)$ it holds $T_j^{\mathbf{i}}(\omega) \subseteq y_j^{\mathbf{i}} + M\varepsilon_j^\beta [-1/2, 1/2]^{N-1}$, where $\beta = (N-1)/(N-p+a)$.

(Hp 4). The sequence $(\delta_j \varepsilon_j^{-N+1})$ has a limit in $[0, +\infty]$. We denote such a value Λ .

Assumptions (Hp 1)-(Hp 3) were introduced in [7] (see Remark 5.4 for a weak variant of (Hp 3)).

In the following remarks we briefly comment on the previous assumptions.

Remark 2.2. *The capacity scaling assumption implies that*

$$\text{cap}_{p,\mu}(T_j(\omega)) \leq \sum_{\mathbf{i} \in \mathbf{Z}^{N-1}} \text{cap}_{p,\mu}(T_j^{\mathbf{i}}(\omega)) = \delta_j \sum_{\mathbf{i} \in \mathbf{Z}^{N-1}} \gamma(\mathbf{i}, \omega).$$

Heuristically, we may assume $\text{cap}_{p,\mu}(T_j(\omega)) \sim \sum_{\mathbf{i} \in \mathbf{Z}^{N-1}} \text{cap}_{p,\mu}(T_j^{\mathbf{i}}(\omega))$ since the obstacles $T_j^{\mathbf{i}}(\omega)$ are sufficiently far apart one from the other by the strong separation assumption. Hence, by taking into account Birkhoff's individual ergodic theorem \mathbb{P} a.s. in Ω we infer

$$\text{cap}_{p,\mu}(T_j(\omega)) \sim \Lambda \mathbb{E}[\gamma].$$

Thus we can distinguish three regimes according to the asymptotic behaviour of $\delta_j \varepsilon_j^{-N+1}$ (see Theorem 2.4).

Remark 2.3. *The stationarity property is a mild assumption in order to have some averaging along the homogenization process, a condition weaker than periodicity or quasi-periodicity. It implies that the random field γ is statistically homogeneous w.r.to the action of traslations compatible with the underlying periodic lattice, e.g. the random variables $\gamma(\mathbf{i}, \cdot)$ are independent and identically distributed.*

With fixed exponents $a \in (-1, +\infty)$ and $p \in ((1+a) \vee 1, N+a)$ (these restrictions will be justified in Section 3, Remark 2.11 and Appendix A), consider the measure $\mu := |x_N|^a d\mathcal{L}^N$ and the corresponding weighted Sobolev space $W^{1,p}(U, \mu)$ (see Section 3).

Let ψ be upper bounded and continuous in the relative interior of $\partial_N U$ w.r.to the relative topology of $\{x_N = 0\}$ (for some comments on this assumption see Remark 2.10) and define the functional $\mathcal{F}_j : L^p(U, \mu) \times \Omega \rightarrow [0, +\infty]$ by

$$\mathcal{F}_j(u, \omega) = \begin{cases} \int_U |\nabla u|^p d\mu & \text{if } u \in W^{1,p}(U, \mu), \tilde{u} \geq \psi \text{ cap}_{p,\mu} \text{ q.e. on } T_j(\omega) \cap \partial_N U \\ +\infty & \text{otherwise.} \end{cases} \quad (2.2)$$

Here, $\text{cap}_{p,\mu}$ is the variational (p, μ) -capacity associated with μ , and \tilde{u} denotes the precise representative of u which is defined except on a set of capacity zero (see Section 3).

To state the main result of the paper and not to make it trivial we also assume that (see Remark 2.9)

(Hp 5). there exists $f \in W^{1,p}(U, \mu)$ such that $\tilde{f} \geq \psi \operatorname{cap}_{p,\mu}$ q.e. on $\partial_N U$.

Theorem 2.4. Assume (Hp 1)-(Hp 5) hold true, $N \geq 2$, and that $a \in (-1, +\infty)$, $p \in ((1+a) \vee 1, N+a)$.

Then there exists a set $\Omega' \subseteq \Omega$ of full probability such that for all $\omega \in \Omega'$ the sequence $(\mathcal{F}_j(\cdot, \omega))_j$ Γ -converges in the $L^p(U, \mu)$ topology to the functional $\mathcal{F} : L^p(U, \mu) \rightarrow [0, +\infty]$ defined by

$$\mathcal{F}(u) = \int_U |\nabla u|^p d\mu + \frac{1}{2} \Lambda \mathbb{E}[\gamma] \int_{\partial_N U} |(\psi(y) - u(y, 0)) \vee 0|^p dy \quad (2.3)$$

if $u \in W^{1,p}(U, \mu)$, $+\infty$ otherwise.

In case U is not bounded equi-coercivity for the functionals \mathcal{F}_j is ensured only in the $L^p_{\text{loc}}(U, \mu)$ topology. A relaxation phenomenon takes place and the domain of the limit has to be slightly enlarged according to Sobolev-Gagliardo-Nirenberg inequality in

$$K^p(U, \mu) = \{u \in L^{p^*}(U, \mu) : \nabla u \in (L^p(U, \mu))^N\}, \quad (2.4)$$

where $p^* = (N+a)p/(N+a-p)$ is the Sobolev exponent relative to $W^{1,p}(\mathbf{R}^N, \mu)$ (see Lemma 3.2). We show Γ -convergence in that case, too

Theorem 2.5. Under the assumptions of Theorem 2.4, if U is unbounded there exists a set $\Omega' \subseteq \Omega$ of full probability such that for all $\omega \in \Omega'$ the family $(\mathcal{F}_j(\cdot, \omega))_j$ Γ -converges in the $L^p_{\text{loc}}(U, \mu)$ topology to the functional $\mathcal{F} : L^p_{\text{loc}}(U, \mu) \rightarrow [0, +\infty]$ defined by

$$\mathcal{F}(u) = \int_U |\nabla u|^p d\mu + \frac{1}{2} \Lambda \mathbb{E}[\gamma] \int_{\partial_N U} |(\psi(y) - u(y, 0)) \vee 0|^p dy \quad (2.5)$$

if $u \in K^p(U, \mu)$, $+\infty$ otherwise.

The set Ω' referred to in the statements of Theorem 2.4, 2.5 is defined in Section 5 below.

Theorem 2.4 is compatible with the addition of boundary data. Assume that U is bounded, denote by Σ a non-empty and relatively open subset of $\partial U \setminus \partial_N U$, and by $W^{1,p}_{0,\Sigma}(U, \mu)$ the strong closure in $W^{1,p}(U, \mu)$ of the restrictions to U of functions $C^\infty(\mathbf{R}^N)$ vanishing on a neighbourhood of $\overline{\Sigma}$. Further, we require that $\overline{\Sigma} \cap \partial_N U = \emptyset$ to avoid additional technicalities.

Corollary 2.6. Assume that U is bounded, and that (Hp 1)-(Hp 4) hold true. With fixed $N \geq 2$, $a \in (-1, +\infty)$, $p \in ((1+a) \vee 1, N+a)$ and $u_0 \in W^{1,p}(U, \mu)$ s.t. $\tilde{u}_0 \geq \psi \operatorname{cap}_{p,\mu}$ q.e. on $\partial_N U$ there exists a set $\Omega'' \subseteq \Omega$ of full probability such that for all $\omega \in \Omega''$ the functionals $\mathcal{F}_j(\cdot, \omega) + \mathcal{X}_{u_0 + W^{1,p}_{0,\Sigma}(U, \mu)}$ Γ -converge in the $L^p(U, \mu)$ topology to $\mathcal{F} + \mathcal{X}_{u_0 + W^{1,p}_{0,\Sigma}(U, \mu)}$, where $\mathcal{X}_{u_0 + W^{1,p}_{0,\Sigma}(U, \mu)}$ is the $0, +\infty$ characteristic function of the subspace $u_0 + W^{1,p}_{0,\Sigma}(U, \mu)$.

Γ -convergence theory then implies convergence of minimizers provided the equi-coercivity of the \mathcal{F}_j 's holds (see Theorem 2.1). That property is ensured by Theorem 8 [23] in case U is bounded, and by Lemma 3.2 below if U is unbounded.

Corollary 2.7. *Under the assumptions of Corollary 2.6 let $g \in L^{(p^*)'}(U, \mu)$, $(p^*)'$ denotes the conjugate exponent of p^* , and $u_j(\cdot, \omega)$ be the minimizer of*

$$\min \left\{ \mathcal{F}_j(u, \omega) - \int_U g u \, d\mu : u \in u_0 + W_{0, \Sigma}^{1,p}(U, \mu) \right\},$$

then (u_j) converges weakly in $W^{1,p}(U, \mu)$ and \mathbb{P} a.s. in Ω to the minimizer of

$$\min \left\{ \mathcal{F}(u) - \int_U g u \, d\mu : u \in u_0 + W_{0, \Sigma}^{1,p}(U, \mu) \right\}.$$

In addition, if $U = \mathbf{R}_+^N$ and $g \in L^{(p^)'}(U, \mu)$ the minimizer $u_j(\cdot, \omega)$ of*

$$\min \left\{ \mathcal{F}_j(u, \omega) - \int_{\mathbf{R}_+^N} g u \, d\mu : u \in W_0^{1,p}(\mathbf{R}_+^N, \mu) \right\},$$

converges locally weakly in $W^{1,p}(\mathbf{R}_+^N, \mu)$ and \mathbb{P} a.s. in Ω to the minimizer of

$$\min \left\{ \mathcal{F}(u) - \int_{\mathbf{R}_+^N} g u \, d\mu : u \in W_0^{1,p}(\mathbf{R}_+^N, \mu) \right\}.$$

Remark 2.8. *Theorem 2.4 recovers the results established in [7] for $p = 2$. Indeed, in the statement there $N \geq 2$, $a \in (-1, 1)$ and thus the compatibility condition between a and p is satisfied. The results contained in [6] can also be inferred by the method below (see Section 6).*

Remark 2.9. *In case U has finite measure (Hp 5) is unnecessary since the constant function $\sup_{\partial_N U} \psi$ satisfies it. In general, (Hp 5) suffices to ensure that $\Gamma\text{-lim inf } \mathcal{F}_j$ is finite in some point, i.e. on f . Actually, from Propositions 5.2, 5.3 below we get $\Gamma\text{-lim}_j \mathcal{F}_j(f) = \mathcal{F}(f)$.*

Remark 2.10. *In [7] the obstacle function ψ is taken to be defined on the whole of U and to be $C^{1,1}(U)$, which clearly implies $\sup_{\partial_N U} \psi(\cdot, 0) < +\infty$ if U is bounded. The latter condition is guaranteed also if $\partial_N U = \mathbf{R}^{N-1}$ since the Γ -limit is finite in some point (see Remark 2.9). Indeed, in such a case it follows that $\psi(\cdot, 0) \vee 0 \in L^p(\mathbf{R}^{N-1})$. More generally, this holds whenever $\partial_N U$ is not quasibounded.*

Remark 2.11. *The restrictions $a > -1$ and $p > 1 + a$ avoid trivial results. Indeed, if $a \leq -1$ or $p \leq 1 + a$ then $W^{1,p}(U, \mu) \equiv W_0^{1,p}(U, \mu)$ (see [25, Proposition 9.10]), and the compatibility condition in (Hp 5) leads to $\psi \leq 0$. Hence, no finite penalization term would appear in the homogenization limit.*

3. SOBOLEV SPACES WITH \mathcal{A}_p WEIGHTS

3.1. Generalities. We recall that a function $w \in L_{\text{loc}}^1(\mathbf{R}^N, (0, +\infty))$ is called a *Muckenhoupt p -weight* for $p \in (1, +\infty)$, in short $w \in \mathcal{A}_p(\mathbf{R}^N)$, if $w \in L_{\text{loc}}^1(\mathbf{R}^N, (0, +\infty))$ and

$$\sup_{r>0, z \in \mathbf{R}^N} \left(r^{-N} \int_{B_r(z)} w(x) \, dx \right) \left(r^{-N} \int_{B_r(z)} w^{1/(1-p)}(x) \, dx \right)^{p-1} < +\infty. \quad (3.1)$$

In the sequel we will consider only weight functions of the form $w(x) = |x_N|^a$, with $a \in (-1, +\infty)$ and $p > (1 + a) \vee 1$ (see the Appendix A and Remark 2.11). Then we define the Radon measure μ on \mathbf{R}^N

by $\mu := wd\mathcal{L}^N$. Take note that $\mathcal{L}^N \ll \mu$ and $\mu \ll \mathcal{L}^N$. If $A \subseteq \mathbf{R}^N$ is an open set the space $H^{1,p}(A, \mu)$ defined as the closure of $C^\infty(A)$ in $L^p(A, \mu)$ under the norm

$$\|\varphi\|_{H^{1,p}(A, \mu)} = \|\varphi\|_{L^p(A, \mu)} + \|\nabla \varphi\|_{(L^p(A, \mu))^N}$$

shares several properties with the usual unweighted case. In particular, Meyers and Serrin's $H = W$ property holds (see [23]). We will give precise references for those properties employed in the sequel in the respective places. We will mainly refer to the book [21], and to [25] when the general theory of weighted Sobolev spaces is concerned. Hereafter we quote explicitly only those results which will be repeatedly used in the proofs below.

Lemma 3.1. *Let $A \subseteq \mathbf{R}^N$ be a connected bounded open set, then for any $E \subseteq A$ with $\mathcal{L}^N(E) > 0$ there exists a constant $c = c(A, E, N, p, \mu) > 0$ such that*

$$\int_{rA} |u - u_{rE}|^p d\mu \leq cr^p \int_{rA} |\nabla u|^p d\mu \quad (3.2)$$

for any $r > 0$ and $u \in W^{1,p}(rA, \mu)$, where $u_{rE} = \int_{rE} u d\mu$.

The (scaled) Poincaré inequality stated above can be inferred by the usual proof by contradiction in case $r = 1$ and a simple scaling argument (see [21, Theorem 1.31] for weak compactness results in weighted Sobolev spaces). Let us then establish a weighted Sobolev-Gagliardo-Nirenberg inequality.

Lemma 3.2. *Let $a \in (-1, +\infty)$, $p \in ((1+a) \vee 1, N+a)$. There exists a constant $c = c(N, p, \mu) > 0$ such that*

$$\|u\|_{L^{p^*}(\mathbf{R}^N, \mu)} \leq c \|\nabla u\|_{(L^p(\mathbf{R}^N, \mu))^N} \quad (3.3)$$

for all $u \in K^p(\mathbf{R}^N, \mu)$, where $p^* = (N+a)p/(N+a-p)$ (see (2.4)).

Proof. Let us first notice that the measure μ is p -admissible according to [21, Chapters 1,5]. By [21, Theorem 15.21] there exist constants $\chi > 1$ and $c = c(N, p, \mu) > 0$ such that

$$\left(\int_{B_r} |u|^{\chi p} d\mu \right)^{1/(\chi p)} \leq cr \left(\int_{B_r} |\nabla u|^p d\mu \right)^{1/p} \quad (3.4)$$

for all $r > 0$ and $u \in C_0^\infty(B_r)$. Being the measure $\mu = |x_N|^a d\mathcal{L}^N$ $(N+a)$ -homogeneous a scaling argument shows that $\chi = p^*/p$. Thus, (3.4) rewrites as (3.3) for all $r > 0$ and $u \in C_0^\infty(B_r)$.

The equality $W^{1,p}(\mathbf{R}^N, \mu) = W_0^{1,p}(\mathbf{R}^N, \mu)$ (see [21, Theorem 1.27]) and (3.4) justify (3.3) for Sobolev maps by a density argument. Eventually, given $u \in K^p(\mathbf{R}^N, \mu)$ let φ_n be a cut-off function between B_n and B_{2n} with $\|\nabla \varphi_n\|_{(L^\infty(\mathbf{R}^N))^N}^p \leq 2/n^p$, we claim that $u_n = \varphi_n u \in W^{1,p}(\mathbf{R}^N, \mu)$ converges strongly to u in $L^{p^*}(\mathbf{R}^N, \mu)$ and ∇u_n converges strongly to ∇u in $(L^p(\mathbf{R}^N, \mu))^N$. Indeed, we have

$$\int_{\mathbf{R}^N} |u_n - u|^{p^*} d\mu \leq \int_{\mathbf{R}^N \setminus B_n} |u|^{p^*} d\mu$$

and by Hölder's inequality

$$\begin{aligned} \int_{\mathbf{R}^N} |\nabla(u_n - u)|^p d\mu &\leq 2^{p-1} \int_{\mathbf{R}^N \setminus B_n} |\nabla u|^p d\mu + \frac{2^p}{n^p} \int_{B_{2n} \setminus B_n} |u|^p d\mu \\ &\leq 2^{p-1} \int_{\mathbf{R}^N \setminus B_n} |\nabla u|^p d\mu + 2^p (\mu(B_2 \setminus B_1))^{p/(N+a)} \left(\int_{B_{2n} \setminus B_n} |u|^{p^*} d\mu \right)^{p/p^*} \end{aligned}$$

and so the conclusion follows. \square

Finally, we recall a trace result in the weighted setting.

Theorem 3.3 (Theorem 9.14 [25], Sec. 10.1 [28]). *Let $A \subseteq \mathbf{R}^N$ be a Lipschitz bounded open set, if $a \in (-1, +\infty)$ and $p > 1 + a$ there exists a compact operator $\text{Tr} : W^{1,p}(A, \mu) \rightarrow L^p(\partial_N A)$ such that $\text{Tr}(u) = u$ for every $u \in C^\infty(\overline{A})$.*

In the rest of the paper to denote the trace of a function $u \in W^{1,p}(A, \mu)$ on $\partial_N A$ we use the more appealing notation $u(\cdot, 0)$.

3.2. Variational (p, μ) -capacities. We recall the notion of variational (p, μ) -capacity (see [21, Chapter 2]): Given any open set $A \subseteq \mathbf{R}^N$ and any set $E \subseteq \mathbf{R}^N$ define

$$\text{cap}_{p,\mu}(E, A) := \inf_{\{A' \text{ open} : A' \supseteq E\}} \inf \left\{ \int_A |\nabla u|^p d\mu : u \in W_0^{1,p}(A, \mu), u \geq 1 \text{ } \mathcal{L}^N \text{ a.e. on } A' \right\},$$

with the usual convention $\inf \emptyset = +\infty$. In case $A = \mathbf{R}^N$, $N \geq 2$, we drop the dependence on A and write only $\text{cap}_{p,\mu}(E)$.

Recall that a property holds $\text{cap}_{p,\mu}$ q.e. if it holds up to a set of $\text{cap}_{p,\mu}$ zero. In particular, any function u in $W^{1,p}(A, \mu)$ has a *precise representative* \tilde{u} defined $\text{cap}_{p,\mu}$ q.e. (see [21, Chapter 4] and [22]). By means of this result the following formula holds (see [21, Corollary 4.13] and the subsequent comments)

$$\text{cap}_{p,\mu}(E, A) = \inf \left\{ \int_A |\nabla u|^p d\mu : u \in W_0^{1,p}(A, \mu), \tilde{u} \geq 1 \text{ } \text{cap}_{p,\mu} \text{ q.e. on } E \right\}. \quad (3.5)$$

Thanks to (3.5) it is easy to show that if A is bounded the minimum problem for the capacity has a unique minimizer $u^{E,A}$, called the (p, μ) -*capacitary potential* of E in A . Instead, in case $A = \mathbf{R}^N$ the minimizer might not exist. The minimum problem has to be relaxed, so that it has a (unique) solution, denoted by u^E , in the space $K^p(\mathbf{R}^N, \mu)$ by Lemma 3.2.

Simple truncation arguments imply that $0 \leq u^E \leq 1$ \mathcal{L}^N a.e. on \mathbf{R}^N , and for every $\lambda > 0$ we get by scaling

$$\text{cap}_{p,\mu}(\lambda E, \lambda A) = \lambda^{N-p+a} \text{cap}_{p,\mu}(E, A). \quad (3.6)$$

For this reason we will restrict ourselves to the range $p < N + a$ to be sure that points have zero capacity (see for instance [21, Theorem 2.19]).

If A and E are symmetric with respect to the hyperplane $x_N = 0$ then the (p, μ) -capacitary potential of E in A , $u^{E,A}$, enjoys the same symmetry and in addition it satisfies

$$\int_{A \cap \mathbf{R}_+^N} |\nabla u^{E,A}|^p d\mu = \frac{1}{2} \text{cap}_{p,\mu}(E, A). \quad (3.7)$$

Moreover, $\text{cap}_{p,\mu}(E + z, A + z) = \text{cap}_{p,\mu}(E, A)$ if $z \in \mathbf{R}^{N-1} \times \{0\}$, being μ unaffected by horizontal translations.

Some further properties are needed. The results below are elementary, but since we have found no explicit reference in literature we prefer to give full proofs.

First we show that set inclusion induces a partial ordering among capacitary potentials.

Proposition 3.4. *Assume $E \subseteq F$, then $u^E \leq u^F$ \mathcal{L}^N a.e. in \mathbf{R}^N .*

Proof. Assume by contradiction that $\mathcal{L}^N(\{u^F < u^E\}) > 0$, then the test-function $\varphi = (u^E - u^F) \vee 0 \in W_0^{1,p}(\mathbf{R}^N, \mu)$ is not identically 0. Notice that

$$\nabla \varphi = \begin{cases} \nabla(u^E - u^F) & \mathcal{L}^N \text{ a.e. in } \{u^F < u^E\} \\ 0 & \mathcal{L}^N \text{ a.e. in } \{u^E \leq u^F\} \end{cases} \quad (3.8)$$

(see [21, Theorem 1.20]). By exploiting the strict minimality of u^F for the capacitary problem related to F , and by comparing its energy with that of $u^F + \varphi$, (3.8) entails

$$\int_{\{u^F < u^E\}} |\nabla u^F|^p d\mu < \int_{\{u^F < u^E\}} |\nabla u^E|^p d\mu. \quad (3.9)$$

Let us now define $w = u^E \wedge u^F$, then w is admissible for the capacitary problem related to E , and by computing its energy we infer from (3.9)

$$\int_{\mathbf{R}^N} |\nabla w|^p d\mu = \int_{\{u^E \leq u^F\}} |\nabla u^E|^p d\mu + \int_{\{u^F < u^E\}} |\nabla u^F|^p d\mu < \int_{\mathbf{R}^N} |\nabla u^E|^p d\mu,$$

which is clearly a contradiction. \square

In turn, Proposition 3.4 yields uniform convergence of the relative capacities to the global one for sets contained in a bounded open given one. In doing that we exploit De Giorgi's slicing-averaging method to refine the cut-off argument contained in Lemma 3.2.

Proposition 3.5. *For any bounded set $E \subset \mathbf{R}^N$ we have*

$$\lim_n \text{cap}_{p,\mu}(E, B_n) = \inf_n \text{cap}_{p,\mu}(E, B_n) = \text{cap}_{p,\mu}(E). \quad (3.10)$$

Furthermore, given a bounded open set $A \subseteq \mathbf{R}^N$, then

$$\lim_n \sup_{\{E: E \subseteq A\}} |\text{cap}_{p,\mu}(E, B_n) - \text{cap}_{p,\mu}(E)| = 0. \quad (3.11)$$

Proof. Assume $E \subset\subset B_m$, and let φ_n^k be a cut-off function between B_{nk} and $B_{n(k+1)}$, $n, r \in \mathbf{N}$ with $n \geq r \geq m$ and $k \in \{1, \dots, r-1\}$, such that $\|\nabla \varphi_n^k\|_{(L^\infty(\mathbf{R}^N))^N}^p \leq 2/n^p$. Thus, it follows for every such k

$$\begin{aligned} \text{cap}_{p,\mu}(E, B_{n(k+1)}) &\leq \int_{\mathbf{R}^N} |\nabla(\varphi_n^k u^E)|^p d\mu \\ &\leq \int_{B_{nk}} |\nabla u^E|^p d\mu + 2^{p-1} \int_{B_{n(k+1)} \setminus B_{nk}} |\nabla u^E|^p d\mu + \frac{2^p}{n^p} \int_{B_{n(k+1)} \setminus B_{nk}} |u^E|^p d\mu. \end{aligned}$$

Hence, by taking into account that $(\text{cap}_{p,\mu}(E, B_i))_{i \in \mathbf{N}}$ is a decreasing sequence and by summing-up on $k \in \{1, \dots, r-1\}$ and averaging we infer

$$\text{cap}_{p,\mu}(E, B_{nr}) \leq \left(1 + \frac{2^p}{r}\right) \text{cap}_{p,\mu}(E) + \frac{2^p}{rn^p} \int_{B_{nr} \setminus B_n} |u^E|^p d\mu.$$

Since $u^E \in L^{p^*}(\mathbf{R}^N, \mu)$ by Lemma 3.2, Hölder's inequality and the $(N+a)$ -homogeneity of μ yield

$$\text{cap}_{p,\mu}(E, B_{nr}) \leq \left(1 + \frac{2^p}{r}\right) \text{cap}_{p,\mu}(E) + 2^p \mu(B_1) r^{p-1} \left(\int_{B_{nr} \setminus B_n} |u^E|^{p^*} d\mu \right)^{p/p^*}. \quad (3.12)$$

In turn, by passing to the limit first as $n \rightarrow +\infty$ and then as $r \rightarrow +\infty$ the latter estimate implies (3.10) being $(\text{cap}_{p,\mu}(E, B_i))_{i \in \mathbf{N}}$ decreasing and bounded from below by $\text{cap}_{p,\mu}(E)$.

Eventually, to get (3.11) notice that with fixed a bounded open set A , $A \subset\subset B_m$, for every $E \subseteq A$ we have $\text{cap}_{p,\mu}(E) \leq \text{cap}_{p,\mu}(A)$ and $0 \leq u^E \leq u^A$ by Proposition 3.4, then (3.12) yields

$$0 \leq \text{cap}_{p,\mu}(E, B_{nr}) - \text{cap}_{p,\mu}(E) \leq \frac{2^p}{r} \text{cap}_{p,\mu}(A) + 2^p \mu(B_1) r^{p-1} \left(\int_{B_{nr} \setminus B_n} |u^A|^{p^*} d\mu \right)^{p/p^*}.$$

By taking into account that $(\text{cap}_{p,\mu}(E, B_i))_{i \in \mathbf{N}}$ is decreasing the uniform convergence is established. \square

4. A WEIGHTED ERGODIC THEOREM

In this section we prove a weighted version of the ergodic theorem relevant in our analysis. We adopt the notation of (Hp 2) and introduce some new. First, take note that $\gamma(\mathbf{i}, \cdot) \in L^\infty(\Omega, \mathbb{P})$ for every $\mathbf{i} \in \mathbf{Z}^{N-1}$, and that the stationarity assumption (2.1) on the τ_i 's yields $\mathbb{E}[\gamma(\mathbf{i}, \cdot)] = \mathbb{E}[\gamma(\mathbf{k}, \cdot)]$ for every $\mathbf{i}, \mathbf{k} \in \mathbf{Z}^{N-1}$, where

$$\mathbb{E}[\gamma(\mathbf{i}, \cdot)] := \int_{\Omega} \gamma(\mathbf{i}, \omega) d\mathbb{P}(\omega).$$

The common value is denoted simply by $\mathbb{E}[\gamma]$. For every $\mathbf{i} \in \mathbf{Z}^{N-1}$ the operator $T_{\mathbf{i}} : L^\infty(\Omega, \mathbb{P}) \rightarrow L^\infty(\Omega, \mathbb{P})$ is defined by $T_{\mathbf{i}}(f) = f \circ \tau_{\mathbf{i}}$. By the stationarity assumption (2.1) it is then easy to check that $\mathcal{S} = \{T_{\mathbf{i}}\}_{\mathbf{i} \in \mathbf{Z}^{N-1}}$ is a multiparameter semigroup generated by the commuting isometries $T_{\mathbf{e}_r}$ for $r \in \{1, \dots, N-1\}$, being $\{\mathbf{e}_1, \dots, \mathbf{e}_{N-1}\}$ the canonical basis of \mathbf{R}^{N-1} .

Theorem 4.1. *Let γ be a process satisfying (Hp 2), then \mathbb{P} a.s. in Ω*

$$\lim_j \frac{1}{\#\mathcal{I}_j(V)} \sum_{\mathbf{i} \in \mathcal{I}_j(V)} \gamma(\mathbf{i}, \omega) = \mathbb{E}[\gamma], \quad (4.1)$$

and

$$\Psi_j(x, \omega) := \sum_{\mathbf{i} \in \mathcal{I}_j(V)} \gamma(\mathbf{i}, \omega) \chi_{Q_j^{\mathbf{i}}}(x) \rightarrow \mathbb{E}[\gamma] \quad \text{weak}^* L^\infty(V). \quad (4.2)$$

for every bounded open set $V \subset \mathbf{R}^{N-1}$ with $\mathcal{L}^{N-1}(\partial V) = 0$.

Proof. With fixed a set V as in the statement above define

$$A_j(f) = \sum_{\mathbf{i} \in \mathbf{Z}^{N-1}} \alpha_{j,\mathbf{i}} T_{\mathbf{i}}(f),$$

where for every $j \in \mathbf{N}$ and $\mathbf{i} \in \mathbf{Z}^{N-1}$ we set $\alpha_{j,\mathbf{i}} = (\#\mathcal{I}_j(V))^{-1} \chi_{\mathcal{I}_j(V)}(\mathbf{i})$. We claim that $(A_j)_{j \in \mathbf{N}}$ is an ergodic \mathcal{S} -net according to [24, p.75], i.e.

- (E1) each A_j is a linear operator on $L^\infty(\Omega, \mathbb{P})$,
- (E2) $A_j(f) \in \overline{\text{co}} \mathcal{S}(f)$ for each $f \in L^\infty(\Omega, \mathbb{P})$ and all $j \in \mathbf{N}$,
- (E3) the A_j 's are equi-continuous, and
- (E4) for each $f \in L^\infty(\Omega, \mathbb{P})$ and $\mathbf{i} \in \mathbf{Z}^{N-1}$

$$\lim_j (A_j(T_{\mathbf{i}}(f)) - A_j(f)) = \lim_j (T_{\mathbf{i}}(A_j(f)) - A_j(f)) = 0.$$

Clearly, (E1) is satisfied. For what (E2) and (E3) are concerned it is enough to notice that $\sum_{\mathbf{i} \in \mathbf{Z}^{N-1}} \alpha_{j,\mathbf{i}} = 1$ for every $j \in \mathbf{N}$. Moreover, for j sufficiently big it holds

$$\begin{aligned} & \sum_{\mathbf{i} \in \mathbf{Z}^{N-1}} \sum_{\{\mathbf{k} \in \mathbf{Z}^{N-1} : |\mathbf{k}|=1\}} |\alpha_{j,\mathbf{i}} - \alpha_{j,\mathbf{i}+\mathbf{k}}| \\ & \leq N \frac{\#\{\mathbf{i} \in \mathbf{Z}^{N-1} : Q_j^{\mathbf{i}} \cap \partial V \neq \emptyset\}}{\#\mathcal{I}_j(V)} \leq N \frac{\mathcal{L}^{N-1}((\partial V)_{\sqrt{N}\varepsilon_j})}{\mathcal{L}^{N-1}(V \setminus (\partial V)_{2\sqrt{N}\varepsilon_j})}. \end{aligned}$$

In turn the latter estimate implies

$$\limsup_j \left(|\alpha_{j,0}| + \sum_{\mathbf{i} \in \mathbf{Z}^{N-1}} \sum_{\{\mathbf{k} \in \mathbf{Z}^{N-1} : |\mathbf{k}|=1\}} |\alpha_{j,\mathbf{i}} - \alpha_{j,\mathbf{i}+\mathbf{k}}| \right) = 0$$

and thus (E4) is satisfied, too. By Eberlein's Theorem (see [24, Theorem 1.5 p.76]) we have that $A_j(f) \rightarrow \bar{f}$ in $L^\infty(\Omega, \mathbb{P})$ for all $f \in L^\infty(\Omega, \mathbb{P})$, where $\bar{f} \in \{g \in \overline{\text{co}} \mathcal{S}(f) : T_{\mathbf{i}}(g) = g \ \forall \mathbf{i} \in \mathbf{Z}^{N-1}\}$. The ergodicity assumption on the $\tau_{\mathbf{i}}$'s implies that \bar{f} is constant \mathbb{P} a.s. in Ω , and since $\sum_{\mathbf{i} \in \mathbf{Z}^{N-1}} \alpha_{j,\mathbf{i}} = 1$ for every $j \in \mathbf{N}$, the convergence $A_j(f) \rightarrow \bar{f}$ in $L^\infty(\Omega, \mathbb{P})$ implies $\bar{f} = \mathbb{E}[f]$.

To deduce (4.1) apply the result above to $\gamma(\underline{0}, \cdot)$ and notice that

$$A_j(\gamma(\underline{0}, \omega)) = \frac{1}{\#\mathcal{I}_j(V)} \sum_{\mathbf{i} \in \mathcal{I}_j(V)} \gamma(\mathbf{i}, \omega)$$

since $\gamma(\mathbf{i}, \omega) = \gamma(\underline{0}, \tau_{\mathbf{i}}(\omega))$ for all $\mathbf{i} \in \mathbf{Z}^{N-1}$ and $\omega \in \Omega$ by (2.1).

Eventually, in order to prove (4.2) consider the family \mathcal{Q} of all open cubes in \mathbf{R}^{N-1} with sides parallel to the coordinate axes, and with center and vertices having rational coordinates. To show the claimed weak* convergence it suffices to check that $\lim_j \int_{\Omega} \Psi_j(x, \omega) \chi_Q(x) d\mathcal{L}^{N-1} = \mathcal{L}^{N-1}(Q) \mathbb{E}[\gamma]$ for any $Q \in \mathcal{Q}$ with $Q \subseteq V$. We have

$$\begin{aligned} & \left| \int_Q (\Psi_j(x, \omega) - \mathbb{E}[\gamma]) d\mathcal{L}^{N-1} \right| \\ & \leq \left| \varepsilon_j^{N-1} \sum_{\mathbf{i} \in \mathcal{I}_j(Q)} \gamma(\mathbf{i}, \omega) - \mathcal{L}^{N-1}(Q) \mathbb{E}[\gamma] \right| + 2\gamma_0 \mathcal{L}^{N-1}(Q \setminus \cup_{\mathbf{i} \in \mathcal{I}_j(Q)} Q_j^{\mathbf{i}}), \end{aligned}$$

and thus (4.1) and the denumerability of \mathcal{Q} yield that the rhs above is infinitesimal \mathbb{P} a.s. in Ω . \square

Remark 4.2. *Even dropping the ergodicity assumption, conclusions similar to those in Theorem 4.1 still hold true. Indeed, by arguing as in the proof above integrating and exploiting the stationarity of γ , the limit $\bar{\gamma}(\underline{Q}, \cdot)$ of the sequence $(A_j(\gamma(\underline{Q}, \cdot)))$ turns out to be characterized as the unique function in $L^\infty(\Omega, \mathbb{P})$ satisfying*

$$\int_I \gamma(\underline{Q}, \omega) d\mathbb{P} = \int_I \bar{\gamma}(\underline{Q}, \omega) d\mathbb{P}$$

for every set $I \in \mathcal{P}$ invariant w.r.to the $\tau_{\mathbf{i}}$'s. Thus, if \mathcal{J} denotes the σ -subalgebra of \mathcal{P} of the invariant sets of the $\tau_{\mathbf{i}}$'s, $\bar{\gamma}(\underline{Q}, \cdot)$ is the conditional expectation of $\gamma(\underline{Q}, \cdot)$ relative to \mathcal{J} , denoted by $\mathbb{E}[\gamma, \mathcal{J}]$. Statement (4.2) then follows analogously.

5. PROOF OF THE MAIN RESULT

Throughout the section the open set $U \subseteq \mathbf{R}_+^N$ will be fixed. Thus, for the sake of simplicity we denote $\mathcal{I}_j := \mathcal{I}_j(\partial_N U)$. Furthermore, $(V_n)_{n \in \mathbf{N}}$ will always denote a sequence of bounded open subsets of $\partial_N U$ with Lipschitz boundary such that $\partial_N U = \cup_n V_n$ and $V_n \subset \subset V_{n+1}$.

The set Ω' mentioned in Theorem 2.4 is defined as any subset of Ω of full probability for which (4.1) and (4.2) hold true for V_n for every $n \in \mathbf{N}$.

In some computations we find inequalities involving constants depending on U , N , p , μ etc... but are always independent from the indexing parameter j . Since it is not essential to distinguish from one specific constant to another, we indicate all of them by the same letter c , leaving understood that c may change from one inequality to another.

Below we prove a joining lemma on varying boundary domains for weighted Sobolev type energies. The argument follows closely that by [1] in the unweighted case for the periodic homogenization on perforated open sets.

Lemma 5.1. *Let (u_j) be converging to u in $L^p(U, \mu)$ for which $\sup_j \|u_j\|_{W^{1,p}(U, \mu)} < +\infty$.*

Let $k \in \mathbf{N}$ and $\omega \in \Omega$ be fixed, then for all $\mathbf{i} \in \mathcal{I}_j$ there exists $h_{\mathbf{i}} \in \{1, \dots, k\}$ such that, having set

$$B_j^{\mathbf{i}, h} := \{x \in U : |x - x_j^{\mathbf{i}}| < 2^{-h} \varepsilon_j\}, \quad C_j^{\mathbf{i}, h} := B_j^{\mathbf{i}, h} \setminus B_j^{\mathbf{i}, h+1},$$

there exists a sequence (v_j) converging to u in $L^p(U, \mu)$ and such that for every $j \in \mathbf{N}$

$$v_j \equiv u_j \text{ on } U \setminus \cup_{\mathbf{i} \in \mathcal{I}_j} C_j^{\mathbf{i}, h_{\mathbf{i}}}, \quad (5.1)$$

$$v_j(x) \equiv (u_j)_{C_j^{\mathbf{i}, h_{\mathbf{i}}}} \text{ if } |x - x_j^{\mathbf{i}}| = \frac{3}{4} 2^{-h_{\mathbf{i}}} \varepsilon_j, \quad x \in U, \quad (5.2)$$

$$\left| \int_A |\nabla v_j|^p d\mu - \int_A |\nabla u_j|^p d\mu \right| \leq \frac{c}{k} \int_{\cup_{\mathbf{i} \in \mathcal{J}_j(A)} Q_j^{\mathbf{i}} \times (0, \varepsilon_j)} |\nabla u_j|^p d\mu \quad (5.3)$$

for some positive constant c independent from j and k , and for all open sets $A \subseteq U$ where

$$\mathcal{J}_j(A) := \{\mathbf{i} \in \mathcal{I}_j : Q_j^{\mathbf{i}} \cap A \neq \emptyset\}.$$

Furthermore, the functions $\zeta_j := \sum_{\mathbf{i} \in \mathcal{I}_j} (u_j)_{C_j^{\mathbf{i}, h_{\mathbf{i}}}} \chi_{Q_j^{\mathbf{i}}}$ converge to u in $L_{\text{loc}}^p(\partial_N U)$.

Proof. For all $j \in \mathbf{N}$, $\mathbf{i} \in \mathcal{I}_j$ and $1 \leq h \leq k$ denote by $\varphi_j^{\mathbf{i}, h}$ a cut-off function between $S_j^{\mathbf{i}, h} := \{x \in U : |x - x_j^{\mathbf{i}}| = \frac{3}{4} 2^{-h} \varepsilon_j\}$ and $U \setminus C_j^{\mathbf{i}, h}$, with $\|\nabla \varphi_j^{\mathbf{i}, h}\|_{(L^\infty(\mathbf{R}^N))^N} \leq 2^{h+2} \varepsilon_j^{-1}$. Then define

$$v_j^{\mathbf{i}, h} := \begin{cases} \varphi_j^{\mathbf{i}, h} (u_j)_{C_j^{\mathbf{i}, h}} + (1 - \varphi_j^{\mathbf{i}, h}) u_j & \text{on } C_j^{\mathbf{i}, h}, \mathbf{i} \in \mathcal{I}_j \\ u_j & \text{otherwise on } U. \end{cases}$$

Being $\text{Lip}(\varphi_j^{\mathbf{i}, h}) 2^{-h-2} \varepsilon_j \leq 1$ we infer

$$\|\nabla v_j^{\mathbf{i}, h}\|_{(L^p(C_j^{\mathbf{i}, h}, \mu))^N}^p \leq \|\nabla u_j\|_{(L^p(C_j^{\mathbf{i}, h}, \mu))^N}^p + \left(\frac{2^{h+2}}{\varepsilon_j} \right)^p \int_{C_j^{\mathbf{i}, h}} |u_j - (u_j)_{C_j^{\mathbf{i}, h}}|^p d\mu,$$

and thus by taking into account Lemma 3.1 we get

$$\|\nabla v_j^{\mathbf{i}, h}\|_{(L^p(C_j^{\mathbf{i}, h}, \mu))^N}^p \leq c \|\nabla u_j\|_{(L^p(C_j^{\mathbf{i}, h}, \mu))^N}^p, \quad (5.4)$$

for some positive constant c depending only on N , p and μ . Indeed, the ratio between the outer and inner radii of $C_j^{\mathbf{i}, h}$ is equals 2 for every \mathbf{i}, j, h .

By summing up and averaging in h , being the $C_j^{\mathbf{i}, h}$ disjoint, we find $h_{\mathbf{i}} \in \{1, \dots, k\}$ such that

$$\|\nabla u_j\|_{(L^p(C_j^{\mathbf{i}, h_{\mathbf{i}}}, \mu))^N}^p \leq \frac{c}{k} \|\nabla u_j\|_{(L^p(Q_j^{\mathbf{i}} \times (0, \varepsilon_j), \mu))^N}^p. \quad (5.5)$$

Define $v_j = v_j^{\mathbf{i}, h_{\mathbf{i}}}$ on $\cup_{\mathbf{i} \in \mathcal{I}_j} C_j^{\mathbf{i}, h_{\mathbf{i}}}$ and $v_j = u_j$ otherwise, then (5.1), (5.2) are satisfied by construction, and (5.3) follows easily from (5.4) and (5.5).

To prove that (v_j) converges to u in $L^p(U, \mu)$ we use again Lemma 3.1. Indeed, by the very definition of v_j we have

$$\begin{aligned} \|u_j - v_j\|_{L^p(U, \mu)} &= \sum_{\mathbf{i} \in \mathcal{I}_j} \|u_j - v_j\|_{L^p(C_j^{\mathbf{i}, h_{\mathbf{i}}}, \mu)} \\ &\leq \sum_{\mathbf{i} \in \mathcal{I}_j} \|u_j - (u_j)_{C_j^{\mathbf{i}, h_{\mathbf{i}}}}\|_{L^p(C_j^{\mathbf{i}, h_{\mathbf{i}}}, \mu)} \leq c \sum_{\mathbf{i} \in \mathcal{I}_j} \frac{\varepsilon_j}{2^{h_{\mathbf{i}}}} \|\nabla u_j\|_{(L^p(C_j^{\mathbf{i}, h_{\mathbf{i}}}, \mu))^N} \leq c \varepsilon_j \|\nabla u_j\|_{(L^p(U, \mu))^N}. \end{aligned}$$

Eventually, let us show the convergence of (ζ_j) to u in $L^p_{\text{loc}}(\partial_N U)$. The (local) compactness of the trace operator (see Theorem 3.3) and the very definition of v_j entail for any open set $V \subset \subset \partial_N U$

$$\limsup_j \|\zeta_j - u\|_{L^p(V)}^p = \limsup_j \|\zeta_j - u_j\|_{L^p(V)}^p \leq \limsup_j \sum_{i \in \mathcal{I}_j} \|u_j - (u_j)_{C_j^{i, h_i}}\|_{L^p(Q_j^i)}^p. \quad (5.6)$$

An elementary scaling argument and the Trace theorem 3.3 yield

$$\begin{aligned} & \|u_j - (u_j)_{C_j^{i, h_i}}\|_{L^p(Q_j^i)}^p \\ & \leq c\varepsilon_j^{-1-a} \left(\|u_j - (u_j)_{C_j^{i, h_i}}\|_{L^p(Q_j^i \times (0, \varepsilon_j), \mu)}^p + \varepsilon_j^p \|\nabla u_j\|_{(L^p(Q_j^i \times (0, \varepsilon_j), \mu))^N}^p \right) \end{aligned} \quad (5.7)$$

for some positive constant c depending only on N , p and μ . Since the scaled Poincaré inequality (3.2) entails

$$\|u_j - (u_j)_{C_j^{i, h_i}}\|_{L^p(Q_j^i \times (0, \varepsilon_j), \mu)}^p \leq c\varepsilon_j^p \|\nabla u_j\|_{(L^p(Q_j^i \times (0, \varepsilon_j), \mu))^N}^p, \quad (5.8)$$

the thesis then follows by collecting (5.6)-(5.8) being $p > 1 + a$. \square

We are now ready to prove the lower bound inequality.

Proposition 5.2. *For all $\omega \in \Omega'$ and $u \in L^p(U, \mu)$*

$$\mathcal{F}(u) \leq \Gamma\text{-}\liminf_j \mathcal{F}_j(u, \omega). \quad (5.9)$$

Proof. We may assume $\Lambda \in (0, +\infty)$, the estimate being trivial if Λ equals 0, while if $\Lambda = +\infty$ it can be inferred by a simple comparison argument with the case Λ finite.

We use the notation introduced in Lemma 5.1, and further set

$$B_j^i := \left\{ x \in \mathbf{R}^N : |x - x_j^i| < \frac{3}{4} 2^{-h_i} \varepsilon_j \right\}, \quad (5.10)$$

for all $i \in \mathcal{I}_j$ (recall that $\mathcal{I}_j = \mathcal{I}_j(\partial_N U)$).

With fixed $u_j \rightarrow u$ in $L^p(U, \mu)$ with $\sup_j \mathcal{F}_j(u_j, \omega) < +\infty$ define the function

$$\xi_j(x) := \begin{cases} (u_j)_{C_j^{i, h_i}} & \text{on } B_j^i \cap U, i \in \mathcal{I}_j \\ v_j(x) & \text{otherwise on } U, \end{cases}$$

where (v_j) is the sequence provided by Lemma 5.1. It is easy to check that $\xi_j \rightarrow u$ in $L^p(U, \mu)$ and $\sup_j \|\xi_j\|_{W^{1,p}(U, \mu)} < +\infty$. By taking into account (5.3) and by splitting the energy contribution of v_j far from and close to the obstacles yields

$$\begin{aligned} & \left(1 + \frac{c}{k}\right) \liminf_j \mathcal{F}_j(u_j, \omega) \geq \liminf_j \mathcal{F}_j(v_j, \omega) \\ & \geq \liminf_j \|\nabla v_j\|_{(L^p(U \setminus \cup_{i \in \mathcal{I}_j} B_j^i, \mu))^N}^p + \liminf_j \sum_{i \in \mathcal{I}_j} \|\nabla v_j\|_{(L^p(B_j^i \cap U, \mu))^N}^p \\ & = \liminf_j \|\nabla \xi_j\|_{(L^p(U, \mu))^N}^p + \liminf_j \sum_{i \in \mathcal{I}_j} \|\nabla v_j\|_{(L^p(B_j^i \cap U, \mu))^N}^p \end{aligned}$$

$$\geq \|\nabla u\|_{(L^p(U,\mu))^N}^p + \liminf_j \sum_{\mathbf{i} \in \mathcal{I}_j} \|\nabla v_j\|_{(L^p(B_j^{\mathbf{i}} \cap U, \mu))^N}^p. \quad (5.11)$$

We claim that for all $\omega \in \Omega'$

$$\liminf_j \sum_{\mathbf{i} \in \mathcal{I}_j} \|\nabla v_j\|_{(L^p(B_j^{\mathbf{i}} \cap U, \mu))^N}^p \geq \frac{1}{2} \Lambda \mathbb{E}[\gamma] \int_{\partial_N U} \Phi(\psi(y) - u(y, 0)) dy, \quad (5.12)$$

where $\Phi(t) := (t \vee 0)^p$. Given this for granted, we infer (5.9) from (5.12) and by letting $k \rightarrow +\infty$ in (5.11).

To conclude we are left with proving (5.12). Denote by $\hat{U} = \text{int}\{(y, x_N) \in \mathbf{R}^N : (y, |x_N|) \in \overline{U}\}$, and extend v_j to \hat{U} by symmetry with respect to the plane $x_N = 0$, i.e. $\hat{v}_j(y, x_N) := v_j(y, |x_N|)$ for $x \in \hat{U}$. Notice that $\hat{v}_j \in W^{1,p}(\hat{U}, \mu)$ and $\|\nabla \hat{v}_j\|_{(L^p(B_j^{\mathbf{i}}, \mu))^N}^p = 2\|\nabla v_j\|_{(L^p(B_j^{\mathbf{i}} \cap U, \mu))^N}^p$. Thus, for every $\mathbf{i} \in \mathcal{I}_j$ we infer by property (5.2) in Lemma 5.1

$$\begin{aligned} \|\nabla v_j\|_{(L^p(B_j^{\mathbf{i}} \cap U, \mu))^N}^p &= \frac{1}{2} \|\nabla \hat{v}_j\|_{(L^p(B_j^{\mathbf{i}}, \mu))^N}^p \\ &\geq \frac{1}{2} \inf \left\{ \|\nabla v\|_{(L^p(B_j^{\mathbf{i}}, \mu))^N}^p : v - (u_j)_{C_j^{\mathbf{i}, h_{\mathbf{i}}}} \in W_0^{1,p}(B_j^{\mathbf{i}}, \mu), \tilde{v} \geq \psi \text{ cap}_{p,\mu} \text{ q.e. on } T_j^{\mathbf{i}}(\omega) \right\} \\ &\geq \frac{1}{2} \inf \left\{ \|\nabla v\|_{(L^p(\mathbf{R}^N, \mu))^N}^p : v \in W_0^{1,p}(\mathbf{R}^N, \mu), \tilde{v} \geq \psi - (u_j)_{C_j^{\mathbf{i}, h_{\mathbf{i}}}} \text{ cap}_{p,\mu} \text{ q.e. on } T_j^{\mathbf{i}}(\omega) \right\}. \end{aligned}$$

With fixed $\eta > 0$ the uniform continuity of ψ on the open set $V_{n+1} \subset \subset \partial_N U$ implies that $\psi(y) \geq \psi(y_j^{\mathbf{i}}) - \eta$ for every $y \in \cup_{\mathbf{i} \in \mathcal{I}_j(V_{n+1})} T_j^{\mathbf{i}}(\omega)$ for j sufficiently big. Thus we deduce

$$\sum_{\mathbf{i} \in \mathcal{I}_j} \|\nabla v_j\|_{(L^p(B_j^{\mathbf{i}}, \mu))^N}^p \geq \frac{1}{2} \delta_j \sum_{\mathbf{i} \in \mathcal{I}_j(V_{n+1})} \gamma(\mathbf{i}, \omega) \Phi(\psi(y_j^{\mathbf{i}}) - (u_j)_{C_j^{\mathbf{i}, h_{\mathbf{i}}}} - \eta). \quad (5.13)$$

In deriving the last inequality we have exploited the p -homogeneity of the weighted norm, formula (3.5), and the capacity scaling assumption in (Hp 1).

To estimate the last term above define $\psi_j := \sum_{\mathbf{i} \in \mathcal{I}_j} (\psi(y_j^{\mathbf{i}}) - (u_j)_{C_j^{\mathbf{i}, h_{\mathbf{i}}}}) \chi_{Q_j^{\mathbf{i}}}$ and consider the functions Ψ_j introduced in Theorem 4.1 for $V = V_{n+1}$, i.e.

$$\Psi_j(y, \omega) = \sum_{\mathbf{i} \in \mathcal{I}_j(V_{n+1})} \gamma(\mathbf{i}, \omega) \chi_{Q_j^{\mathbf{i}}}(y).$$

Recall that by the very definition of Ω' we have $\Psi_j(\cdot, \omega) \rightarrow \mathbb{E}[\gamma]$ weak* $L^\infty(V_{n+1})$ for all $\omega \in \Omega'$. Being $V_n \subset \subset V_{n+1}$, (5.13) rewrites for j sufficiently big as

$$\sum_{\mathbf{i} \in \mathcal{I}_j} \|\nabla v_j\|_{(L^p(B_j^{\mathbf{i}}, \mu))^N}^p \geq \frac{1}{2} \delta_j \varepsilon_j^{-N+1} \int_{V_n} \Phi(\psi_j(y) - \eta) \Psi_j(y, \omega) dy. \quad (5.14)$$

Notice that $\psi_j \rightarrow (\psi - u)$ in $L^p(V_{n+1})$ by the continuity of ψ and by Lemma 5.1. In turn this implies $\Phi(\psi_j - \eta) \rightarrow \Phi(\psi - u - \eta)$ in $L^1(V_{n+1})$ for every $\eta > 0$. Hence, for every $k \in \mathbf{N}$ and $\eta > 0$ we get

$$\liminf_j \sum_{\mathbf{i} \in \mathcal{I}_j} \|\nabla v_j\|_{(L^p(B_j^{\mathbf{i}}, \mu))^N}^p \geq \frac{1}{2} \Lambda \mathbb{E}[\gamma] \int_{V_n} \Phi(\psi(y) - u(y, 0) - \eta) dy.$$

To recover (5.12) let $\eta \rightarrow 0^+$, and then increase V_n to $\partial_N U$. \square

In the next proposition we show that the lower bound derived in Proposition 5.2 is sharp.

Proposition 5.3. *For all $\omega \in \Omega'$ and $u \in L^p(U, \mu)$*

$$\Gamma\text{-}\limsup_j \mathcal{F}_j(u, \omega) \leq \mathcal{F}(u). \quad (5.15)$$

Proof. Let us show that for every $u \in L^p(\Omega, \mu)$ such that $\mathcal{F}(u) < +\infty$ and for every event $\omega \in \Omega'$ we may construct $u_j \in W^{1,p}(U, \mu)$ such that $u_j \rightarrow u$ in $L^p(U, \mu)$ and

$$\limsup_j \mathcal{F}_j(u_j, \omega) \leq \mathcal{F}(u). \quad (5.16)$$

Take note that we may assume $\Lambda \in (0, +\infty)$. Indeed, if $\Lambda = 0$ we may use a comparison argument with the former case to conclude. Instead, if $\Lambda = +\infty$ by Proposition 5.2 we have $\tilde{u} \geq \psi \operatorname{cap}_{p,\mu} \text{ q.e. on } \partial_N U$, and then we may take $u_j \equiv u$.

Furthermore, we may reduce to $u \in C^{0,1} \cap L^\infty(U)$, and $\psi \in L^\infty(\partial_N U)$ and continuous in the relative interior of $\partial_N U$ w.r.to the relative topology of $\{x_N = 0\}$.

Indeed, suppose (5.16) proven under those assumptions. The functions $\psi_k := \psi \vee (-k)$, $k \in \mathbf{N}$, are bounded and continuous on the relative interior of $\partial_N U$, $\psi_k \geq \psi_{k+1} \geq \psi$ and (ψ_k) converges to ψ pointwise. Denote by $\mathcal{F}_j^{\psi_k}$, \mathcal{F}^{ψ_k} the functionals defined as \mathcal{F}_j , \mathcal{F} in (2.2) and (2.5), respectively, with ψ substituted by ψ_k . Clearly, we have $\mathcal{F}_j \leq \mathcal{F}_j^{\psi_k}$, so that $\Gamma\text{-}\limsup_j \mathcal{F}_j(u, \omega) \leq \Gamma\text{-}\limsup_j \mathcal{F}_j^{\psi_k}(u, \omega) = \mathcal{F}^{\psi_k}(u)$. Moreover, notice that $\mathcal{F}^{\psi_k}(u) \rightarrow \mathcal{F}(u)$ as $k \rightarrow +\infty$ being $u \in L^\infty(U)$.

It is easy to check that if $\mathcal{F}(u) < +\infty$ then $\mathcal{F}(u \vee (-k) \wedge k) < +\infty$ for any $k \in \mathbf{N}$ with $k \geq \|\psi\|_{L^\infty(\partial_N U)}$ (see [21, Lemma 1.19] for the fact that truncations preserve $W^{1,p}(U, \mu)$ regularity). The density of $C^{0,1} \cap L^\infty(U)$ in $W^{1,p}(U, \mu)$ and the lower semicontinuity of $\Gamma\text{-}\limsup_j \mathcal{F}_j$ then establish (5.16) for functions in $W^{1,p}(U, \mu)$ once it has been proven for their truncations (see [11, Theorem 1.1] and [23, Theorem 4] for extension and density results in \mathbf{R}^N , respectively).

Clearly, if ψ is bounded we may also take the function f in (Hp 5) to be in $L^\infty(U)$ upon substituting it with its truncation at the levels $\pm \|\psi\|_{L^\infty(\partial_N U)}$.

To conclude the proof we distinguish two cases according to whether U is bounded or not.

Step 1: U is bounded. With fixed $\eta > 0$ such that

$$\mathcal{H}^{N-1}(\{y \in \partial_N U : \psi(y) - u(y, 0) = \eta\}) = 0, \quad (5.17)$$

consider the (relatively) open sets $\Sigma := \{y \in \partial_N U : u(y, 0) + \eta < \psi(y)\}$, $\Sigma_n := \Sigma \cap V_n$, and the set of indexes $\mathcal{J}_j := \{i \in \mathbf{Z}^{N-1} : Q_j^i \cap \Sigma \neq \emptyset\}$. By the uniform continuity of ψ on V_n we have $\psi(y) \leq \psi(y_j^i) + \eta$ for every $y \in \cup_{i \in \mathcal{J}_j} Q_j^i$ for j sufficiently big. Set $\lambda_j := \delta_j^{1/(N-p+a)}$, define $\tilde{T}_j^i(\omega) := (T_j^i(\omega) - y_j^i)/\lambda_j$ and notice that $\tilde{T}_j^i(\omega) \subseteq B_{m-1}$ for some $m \in \mathbf{N}$ by (Hp 3). Then (3.11) in Proposition 3.5 yields

$$\sup_{i \in \mathbf{Z}^{N-1}} |\operatorname{cap}_{p,\mu}(\tilde{T}_j^i(\omega), B_n) - \operatorname{cap}_{p,\mu}(\tilde{T}_j^i(\omega))| \leq \eta \quad (5.18)$$

for all $n > m$ large enough. Let $\xi_j^i \in W_0^{1,p}(B_n, \mu)$ be such that $\tilde{\xi}_j^i \geq 1 \text{ cap}_{p,\mu}$ q.e. $\tilde{T}_j^i(\omega)$ and $\text{cap}_{p,\mu}(\tilde{T}_j^i(\omega), B_n) = \|\nabla \xi_j^i\|_{(L^p(B_n, \mu))^N}^p$, and let $\zeta \in C_0^\infty(B_m)$ be any function such that $\zeta \equiv 1$ on B_{m-1} , $\|\nabla \zeta\|_{(L^\infty(B_m))^N}^p \leq 2$ and $0 \leq \zeta \leq 1$.

With fixed $n \in \mathbf{N}$ for which (5.18) holds, let (v_j) be the sequence provided by Lemma 5.1 with $u_j \equiv u + \eta$ and $k = n$. Define

$$u_j(x) := \begin{cases} \left(1 - \xi_j^i \left(\frac{x - x_j^i}{\lambda_j}\right)\right) (u + \eta)_{C_j^{i,h_1}} + \xi_j^i \left(\frac{x - x_j^i}{\lambda_j}\right) (\psi(y_j^i) + \eta) & U \cap B_j^i, i \in \mathcal{I}_j(\Sigma_n) \\ v_j(x) & U \setminus \cup_{\mathcal{J}_j} B_j^i \\ \left(1 - \zeta \left(\frac{x - x_j^i}{\lambda_j}\right)\right) v_j(x) + \zeta \left(\frac{x - x_j^i}{\lambda_j}\right) f(x) & U \cap B_{m\lambda_j}(x_j^i), i \in \mathcal{J}_j \setminus \mathcal{I}_j(\Sigma_n). \end{cases} \quad (5.19)$$

In the definition above B_j^i is the set defined in (5.10), and $f \in W^{1,p}(U, \mu) \cap L^\infty(U)$ is as in (Hp 5).

Take note that $u_j \rightarrow u + \eta$ in $L^p(U, \mu)$ since $\mathcal{L}^N(\{u_j \neq u + \eta\}) \rightarrow 0$ and U, u, f, ψ are bounded. Clearly, $\tilde{u}_j \geq \psi \text{ cap}_{p,\mu}$ q.e. on $T_j(\omega)$, and then by the choice of ξ_j^i , (3.6), (3.7) and (5.18) give

$$\lambda_j^{-p} \int_{\mathbf{R}_+^N \cap B_j^i} \left| \nabla \xi_j^i \left(\frac{x - x_j^i}{\lambda_j} \right) \right|^p d\mu = \frac{\delta_j}{2} \text{cap}_{p,\mu}(\tilde{T}_j^i(\omega), B_n) \leq \frac{1}{2} (\text{cap}_{p,\mu}(T_j^i(\omega)) + \delta_j \eta)$$

for all $i \in \mathbf{Z}^{N-1}$. An analogous formula holds for the translated and scaled gradient of ζ . Thus, a straightforward calculation implies

$$\begin{aligned} \mathcal{F}_j(u_j, \omega) &\leq \int_U |\nabla v_j|^p d\mu + \frac{1}{2} \sum_{i \in \mathcal{I}_j(\Sigma_n)} \Phi(\psi(y_j^i) - u_{C_j^{i,h_1}} + \eta) (\text{cap}_{p,\mu}(T_j^i(\omega)) + \delta_j \eta) \\ &\quad + 2^{p-1} \|u - f\|_{L^\infty(U)}^p \|\nabla \zeta\|_{L^p(B_m)}^p \delta_j \#(\mathcal{J}_j \setminus \mathcal{I}_j(\Sigma_n)) + 2^{p-1} \int_{D_j^n} |\nabla(u - f)|^p d\mu \end{aligned} \quad (5.20)$$

where $\Phi(t) = (t \vee 0)^p$ and $D_j^n = \cup_{\mathcal{J}_j \setminus \mathcal{I}_j(\Sigma_n)} (U \cap B_{m\lambda_j}(x_j^i))$. By taking into account that $\mathcal{L}^N(D_j^n) \rightarrow 0$ as $j \rightarrow +\infty$ we may argue as in Proposition 5.2 to obtain

$$\begin{aligned} \Gamma\text{-}\limsup_j \mathcal{F}_j(u + \eta, \omega) &\leq \int_U |\nabla u|^p d\mu + \frac{c}{n} \\ &\quad + \frac{1}{2} \Lambda(\mathbb{E}[\gamma] + \eta) \int_{V_n} \Phi(\psi(y) - u(y, 0) + \eta) dy + 2^p \|u - f\|_{L^\infty(U)}^p \Lambda \mathcal{H}^{N-1}(\Sigma \setminus \Sigma_n). \end{aligned}$$

Since $\mathcal{H}^{N-1}(\partial\Sigma \cup \partial\Sigma_n) = 0$, by increasing V_n to $\partial_N U$ we get

$$\Gamma\text{-}\limsup_j \mathcal{F}_j(u + \eta, \omega) \leq \int_U |\nabla u|^p d\mu + \frac{1}{2} \Lambda(\mathbb{E}[\gamma] + \eta) \int_{\partial_N U} \Phi(\psi(y) - u(y, 0) + \eta) dy. \quad (5.21)$$

To conclude take note that $(\psi - u) \vee 0 \in L^p(\partial_N U)$ since $\mathcal{F}(u) < +\infty$, then there exists a positive infinitesimal sequence (η_k) such that $\mathcal{H}^{N-1}(\{y \in \partial_N U : \psi(y) - u(y, 0) = \eta_k\}) = 0$ for every $k \in \mathbf{N}$. Moreover, $u + \eta_k \in W^{1,p}(U, \mu)$, being U bounded, and it satisfies (5.21). The thesis then follows by the lower semicontinuity of $\Gamma\text{-}\limsup_j \mathcal{F}_j$ as the rhs of (5.21) converges to $\mathcal{F}(u)$ as $k \rightarrow +\infty$.

Step 2: U unbounded. To remove the boundedness assumption on U we localize the problem: for any open subset A of U , $\omega \in \Omega$ we denote by $\mathcal{F}_j(\cdot, \omega; A)$ and $\mathcal{F}(\cdot; A)$ the functionals defined on $W^{1,p}(U, \mu)$ as \mathcal{F}_j and \mathcal{F} , respectively, with the domain of integration U substituted with A . Consider

an increasing sequence $(U_r)_{r \in \mathbf{N}}$ of bounded open Lipschitz sets in U such that $\cup_r U_r = U$, $B_r \cap U \subseteq U_r \subseteq B_{r+1} \cap U$, and denote by $(V_n^r)_{n \in \mathbf{N}}$ a family of open Lipschitz subsets of U_r such that $V_n^r \subset \subset V_{n+1}^r$ with $\cup_n V_n^r = U_r$. Let φ_r be a cut-off function between B_r and B_{2r} with $\|\nabla \varphi_r\|_{(L^\infty(\mathbf{R}^N))^N}^p \leq 2/r^p$.

We fix $\eta > 0$ for which (5.17) holds true and repeat for each U_r the construction of Step 1. Further, we join the sequence defined as in (5.19) on U_r with the function f on $\mathbf{R}^N \setminus \overline{U_{2r}}$. The sequence obtained with this construction gives the limsup inequality up to a vanishing error.

More precisely, with fixed $r \in \mathbf{N}$ and $n \in \mathbf{N}$ such that (5.18) holds, let (u_j^r) be defined as in (5.19) with Σ and Σ_n substituted by $\Sigma \cap U_{2r}$ and $\Sigma \cap V_n^{2r}$, respectively. Then $(u_j^r) \subset W^{1,p}(U_{2r}, \mu)$ and (u_j^r) converges to $u + \eta$ in $L^p(U_{2r}, \mu)$. Define $w_j^r = \varphi_r u_j^r + (1 - \varphi_r)f$, where $f \in W^{1,p}(U, \mu) \cap L^\infty(U)$ is as in (Hp 5). Take note that $w_j^r \in W^{1,p}(U, \mu)$, $(w_j^r)_j$ converges to $(u + \eta)_r := \varphi_r(u + \eta) + (1 - \varphi_r)f$ in $L^p(U, \mu)$ and by definition $\tilde{w}_j^r \geq \psi \text{ cap}_{p,\mu}$ q.e. on $\partial_N U$. Furthermore, we have

$$\begin{aligned} \mathcal{F}_j(w_j^r, \omega) &\leq \mathcal{F}_j(u_j^r, \omega; U_{2r}) + \mathcal{F}_j(f, \omega; U \setminus \overline{B_{2r}}) \\ &\quad + 2^{p-1} (\mathcal{F}_j(u_j^r, \omega; U_{2r} \setminus \overline{B_r}) + \mathcal{F}_j(f, \omega; U_{2r} \setminus \overline{B_r})) + \frac{2^p}{r^p} \int_{U_{2r} \setminus \overline{B_r}} |u_j^r - f|^p d\mu. \end{aligned}$$

To estimate the rhs above we notice that by (5.3) in Lemma 5.1 the first and third terms can be dealt with as in (5.20). By passing to the limsup first as $j \rightarrow +\infty$, and then for $n \rightarrow +\infty$ we get as in (5.21)

$$\begin{aligned} \Gamma\text{-lim sup}_j \mathcal{F}_j((u + \eta)_r, \omega) &\leq (1 + \eta) \mathcal{F}(u + \eta; U_{2r}) + \int_{U \setminus \overline{B_{2r}}} |\nabla f|^p d\mu \\ &\quad + 2^{p-1} \left((1 + \eta) \mathcal{F}(u + \eta; U_{2r} \setminus \overline{B_r}) + \int_{U_{2r} \setminus \overline{B_r}} |\nabla f|^p d\mu \right) + \frac{2^p}{r^p} \int_{U_{2r} \setminus \overline{B_r}} |u + \eta - f|^p d\mu. \end{aligned} \quad (5.22)$$

Arguing as in Step 1, we choose a positive infinitesimal sequence (η_k) for which (5.17) holds, and since $((u + \eta_k)_r)$ converges to u_r in $L^p(U, \mu)$ as $k \rightarrow +\infty$, the lower semicontinuity of $\Gamma\text{-lim sup}_j \mathcal{F}_j$ and (5.22) yield

$$\begin{aligned} \Gamma\text{-lim sup}_j \mathcal{F}_j(u_r, \omega) &\leq \mathcal{F}(u; U_{2r}) + \int_{U \setminus \overline{B_{2r}}} |\nabla f|^p d\mu \\ &\quad + 2^{p-1} \left(\mathcal{F}(u; U_{2r} \setminus \overline{B_r}) + \int_{U_{2r} \setminus \overline{B_r}} |\nabla f|^p d\mu \right) + \frac{2^p}{r^p} \int_{U_{2r} \setminus \overline{B_r}} |u - f|^p d\mu. \end{aligned} \quad (5.23)$$

Finally, being the rhs in the inequality above a finite measure, the lower semicontinuity of $\Gamma\text{-lim sup}_j \mathcal{F}_j$ gives the conclusion as $r \rightarrow +\infty$ since (u_r) converges to u in $L^p(U, \mu)$. \square

Remark 5.4. *The strong separation assumption in (Hp 3) ensures that the scaled obstacle sets $(T_j^1(\omega) - y_j^1)/\varepsilon_j^{(N-1)/(N-p+a)}$ are equi-bounded and located in small neighbourhoods of the x_j^1 's. It turns out from the proof of Theorem 2.4 (see Propositions 5.2 and 5.3) that this condition can be relaxed into*

(Hp 3)' *There exist $\varepsilon > 0$ and $\beta \in (1, (N-1)/(N-p+a)]$ such that for all $\mathbf{i} \in \mathbf{Z}^{N-1}$, $\omega \in \Omega$, and $\varepsilon_j \in (0, \varepsilon)$ the sets $(T_j^{\mathbf{i}}(\omega) - z_j^{\mathbf{i}}(\omega))/\varepsilon_j^{(N-1)/(N-p+a)}$ are contained in a fixed bounded set, for some points $z_j^{\mathbf{i}}(\omega) \in Q_j^{\mathbf{i}}$, and $T_j^{\mathbf{i}}(\omega) \subseteq y_j^{\mathbf{i}} + \varepsilon_j^\beta [-1/2, 1/2]^{N-1}$.*

The latter condition with $\beta > 1$ is needed to ensure the validity of the joining Lemma 5.1 also in this framework. Instead, the first condition is used when applying Proposition 3.5 in the construction of the recovery sequence in Proposition 5.3.

Remark 5.5. *It is clear from Remark 4.2, Propositions 5.2 and 5.3 that Theorem 2.4 still holds even dropping the ergodicity assumption on the $\tau_{\mathbf{i}}$'s. The Γ -limit $\mathcal{F} : L^p(U, \mu) \times \Omega \rightarrow [0, +\infty]$ being then defined as the functional in (2.5) with $\mathbb{E}[\gamma]$ replaced by the conditional expectation $\mathbb{E}[\gamma, \mathcal{I}]$.*

Slightly refining the argument in Step 2 above we extend the convergence result to the L_{loc}^p topology for unbounded domains.

Proof (of Theorem 2.5). We keep using the notation introduced in Step 2 of Proposition 5.3. The extension result in [11, Theorem 1.1] and Lemma 3.2 ensure that $K^p(U, \mu)$ is the domain of any Γ -cluster point. Furthermore, the liminf inequality easily follows by applying Proposition 5.2 to the localized functionals $\mathcal{F}_j(\cdot, \omega; U_r)$, and then by taking the limit as $r \rightarrow +\infty$.

Instead, to get the limsup inequality for any $u \in K^p(U, \mu)$ we use the sequence constructed in Step 2 of Proposition 5.3 and repeat the same arguments up to (5.23). To this aim take note that $K^p(U, \mu) \subset W_{\text{loc}}^{1,p}(U, \mu)$. Eventually, Hölder's inequality yields

$$\frac{2^p}{r^p} \int_{U_{2r} \setminus \overline{B_r}} |u - f|^p d\mu \leq 2^p (\mu(B_2 \setminus B_1))^{p/(N+a)} \left(\int_{U_{2r} \setminus \overline{B_r}} |u - f|^{p^*} d\mu \right)^{p/p^*},$$

and thus the conclusion follows as in Proposition 5.3 by taking the limit for $r \rightarrow +\infty$ in (5.23). \square

Let us now prove Corollary 2.6.

Proof (of Corollary 2.6). The set Ω'' referred to in the statement is defined analogously to Ω' . Hence, being $u_0 + W_{0,\Sigma}^{1,p}(U, \mu)$ weakly closed, thanks to Proposition 5.2 for all $\omega \in \Omega''$ we have

$$\Gamma\text{-}\liminf_j (\mathcal{F}_j + \mathcal{X}_{u_0 + W_{0,\Sigma}^{1,p}(U, \mu)})(u, \omega) \geq (\mathcal{F} + \mathcal{X}_{u_0 + W_{0,\Sigma}^{1,p}(U, \mu)})(u).$$

Thus, given $u \in u_0 + W_{0,\Sigma}^{1,p}(U, \mu)$ to conclude it suffices to verify that the construction of the sequence (u_j) in Proposition 5.3 with f there substituted by u_0 matches also the boundary condition since $\overline{\Sigma} \cap \partial_N U = \emptyset$. \square

6. GENERALIZATIONS

In the previous sections we have described the asymptotic behaviour of the weighted norms on open sets U subject to an obstacle condition on part of the boundary of U . In the present we discuss some generalizations of Theorem 2.4. We limit ourselves to state the results in these settings, since their proofs follow straightforward from the arguments of Section 5 (see also [1]).

First, we point out that we have treated the case of the p -weighted norm only for the sake of simplicity. Indeed, under only minor changes in the proofs the same results hold for p -homogeneous energy densities. Instead, the extension to non-linear energy densities having p -growth seems to be more difficult. The non-linear capacitary formula introduced by Ansini and Braides [1] in the deterministic setting is related to the geometry of the scaled obstacle set. On the other hand, (Hp 1) involves only the scaling properties of the capacity of the obstacle set, then we are led to formulate (Hp 1)' below.

With fixed any $H : \mathbf{R}^N \rightarrow [0, +\infty)$ and $t \geq 0$ define the H -capacity of a set $E \subseteq \mathbf{R}^N$ by

$$\text{cap}_{H,\mu}(t, E) := \inf_{\{A \text{ open} : A \supseteq E\}} \inf \left\{ \int_{\mathbf{R}^N} H(Du) d\mu : u \in W^{1,p}(\mathbf{R}^N, \mu), u \geq t \mathcal{L}^N \text{ a.e. on } A \right\}.$$

Let $h : \mathbf{R}^N \rightarrow [0, +\infty)$ be a convex function such that

$$c_1(|x|^p - 1) \leq h(x) \leq c_2(|x|^p + 1)$$

for some constants $c_1, c_2 > 0$, and $h(y, x_N) = h(y, -x_N)$ for any $x \in \mathbf{R}^N$. Furthermore, put $H_j(x) := \varepsilon_j^{\frac{(N-1)p}{N-p+a}} h\left(\varepsilon_j^{-\frac{(N-1)}{N-p+a}} x\right)$ for all $x \in \mathbf{R}^N$. A natural and compatible generalization of (Hp 1) is then

(Hp 1)'. Capacitary Scaling: There exist a positive infinitesimal sequence $(\delta_j)_j$, a function $\Phi \in C^0(\mathbf{R}^+)$ and a process $\gamma : \mathbf{Z}^{N-1} \times \Omega \rightarrow [0, +\infty)$ such that for all $\mathbf{i} \in \mathbf{Z}^{N-1}$ and $\omega \in \Omega$ we have

$$\lim_j \text{cap}_{H_j,\mu} \left(t, (T_j^{\mathbf{i}}(\omega) - y_j^{\mathbf{i}}) / \delta_j^{1/(N-p+a)} \right) = \Phi(t) \gamma(\mathbf{i}, \omega).$$

Indeed, in case h is p -homogeneous we have $H_j \equiv h$, $\text{cap}_{H_j,\mu}(t, E) = t^p \text{cap}_{h,\mu}(1, E)$, and thus we may take $\Phi(t) = t^p$. In the fully deterministic setting, i.e. $T_j^{\mathbf{i}}(\omega) = y_j^{\mathbf{i}} + \delta_j^{1/(N-p+a)} T$ for some $T \subseteq \mathbf{R}^{N-1}$ for all $\omega \in \Omega$, $\mathbf{i} \in \mathbf{Z}^{N-1}$ and $j \in \mathbf{N}$, by assuming that $(H_j)_j$ converges pointwise to H (this holds upon extracting a subsequence by the growth conditions of h), we have $\lim_j \text{cap}_{H_j,\mu}(t, T) = \text{cap}_{H,\mu}(t, T)$ (see [5, Proposition 12.8]). The continuity of $\text{cap}_{H,\mu}(\cdot, T)$ holds thanks to the local equi-Lipschitz continuity of the H_j 's (which is a consequence of their convexity and the growth conditions of h).

Next we define the functional $\mathcal{F}_j^h : L^p(U, \mu) \times \Omega \rightarrow [0, +\infty]$ by

$$\mathcal{F}_j^h(u, \omega) = \begin{cases} \int_U h(\nabla u) d\mu & \text{if } u \in W^{1,p}(U, \mu), \tilde{u} \geq \psi \text{ cap}_{p,\mu} \text{ q.e. on } T_j(\omega) \cap \partial_N U \\ +\infty & \text{otherwise.} \end{cases} \quad (6.1)$$

The arguments by [1] and those of Section 5 then give the following result.

Theorem 6.1. *Assume (Hp 1)' and (Hp 2)-(Hp 5) hold true, $N \geq 2$, and that $a \in (-1, +\infty)$, $p \in ((1+a) \vee 1, N+a)$.*

Then there exists a set $\Omega' \subseteq \Omega$ of full probability such that for all $\omega \in \Omega'$ the family $(\mathcal{F}_j^h(\cdot, \omega))_j$ Γ -converges in the $L^p(U, \mu)$ topology to the functional $\mathcal{F}^h : L^p(U, \mu) \rightarrow [0, +\infty]$ defined by

$$\mathcal{F}^h(u) = \int_U h(\nabla u) d\mu + \frac{1}{2} \Lambda \mathbb{E}[\gamma] \int_{\partial_N U} \Phi((\psi(y) - u(y, 0)) \vee 0) dy \quad (6.2)$$

if $u \in W^{1,p}(U, \mu)$, $+\infty$ otherwise.

Eventually, let us point out that similar results hold also in case the obstacles are periodically equi-distributed inside the open set U . Clearly, conditions (Hp 1)-(Hp 5) have to be reformulated in order to deal with the N -dimensional setting. The analogue of Theorem 6.1 is then an easy consequence of the arguments of Section 5 and those by [1].

In particular, the homogenization results in perforated open sets by [1] can be extended to the ergodic setting of Section 5, thus recovering the results of [6], too.

APPENDIX A.

We show that $w(x) = |x_N|^a$ belongs to the Muckenhoupt class $\mathcal{A}_p(\mathbf{R}^N)$ under a compatibility condition between p and a .

Lemma A.1. *For $p > (1+a) \vee 1$ and $a > -1$ the function $w(x) = |x_N|^a$ is in the Muckenhoupt's class $\mathcal{A}_p(\mathbf{R}^N)$ (see (3.1) for the definition).*

Proof. Let us first point out that conditions $a > -1$ and $p > 1+a$ are imposed only to guarantee the local integrability of w and $w^{1/(1-p)}$, respectively.

Being $w = w(x_N)$ and even, we may restrict the supremum in (3.1) to points $z = (0, z_N)$ with $z_N \geq 0$. Define $I_\alpha(r, z_N) := \int_{B_r(z)} |x_N|^\alpha dx$ for $\alpha > -1$. A direct integration yields

$$I_\alpha(r, z_N) \leq 3^{|\alpha|} \omega_N r^N \left(r \vee \frac{z_N}{2} \right)^\alpha, \quad (\text{A.1})$$

for $\alpha \geq 0$ and for $\alpha \in (-1, 0)$ provided $z_N \geq 2r$. Instead, in case $\alpha \in (-1, 0)$ and $z_N \in (0, 2r)$ we have $B_r(z) \subseteq B_{3r}(\underline{0})$ and again by a direct integration we deduce

$$I_\alpha(r, z_N) \leq \frac{2N\omega_N}{1+\alpha} \left(\int_0^1 (1-t^2)^{(1+\alpha)/2} dt \right) (3r)^{N+\alpha}. \quad (\text{A.2})$$

In any case, by applying estimates (A.1) and (A.2) above we infer

$$I_a(r, z_N) \left(I_{-\frac{a}{p-1}}(r, z_N) \right)^{p-1} \leq c r^{Np}$$

for some positive constant $c = c(N, a, p)$. Clearly, this is equivalent to (3.1). \square

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